MA 441, Fall 2015

Assignment 9

This homework is due *Thursday* Nov 5.

There are total 21 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers section 4.2, 5.1 in Bartle–Sherbert.

(1) [2pt] (Theorem 4.2.4 for difference) *Using ε–δ definition,* prove that limit of functions preserves difference. That is, prove the following:
Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A, and f, g be functions on A to $\mathbb{R}$. If $\lim\limits_{x\to c} f = L$, and $\lim\limits_{x\to c} g = M$, then $\lim\limits_{x\to c} f - g = L - M$.

(2) [3pt] Using arithmetic properties of limit, find the following limits.
  (a) $\lim\limits_{x\to 1} \frac{x^{100}+2}{x^{100}-2}$.
  (b) $\lim\limits_{x\to 0} \frac{(x+1)^{20}-1}{x}$.
  (c) $\lim\limits_{x\to c} \frac{(x-c+1)^{2}-1}{x-c}$.

(3) (a) [2pt] (4.2.5) Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim\limits_{x\to c} g = 0$. Prove that $\lim\limits_{x\to c} fg = 0$.
Explain why Theorem 4.2.4 (Arithmetic Properties of Limit) cannot be used.
  (b) [1pt] ($\sim$4.2.11b) Determine whether $\lim\limits_{x\to 0} x\cos(1/x^2)$ exists in $\mathbb{R}$.

(4) [2pt] (4.2.13) Functions f and g are defined on $\mathbb{R}$ by $f(x) = x + 1$ and $g(x) = 2$ if $x \neq 1$ and $g(1) = 0$.
  (a) Find $\lim\limits_{x\to 1} g(f(x))$ and compare with the value of $g(\lim\limits_{x\to 1} f(x))$.
  (b) Find $\lim\limits_{x\to 1} f(g(x))$ and compare with the value of $f(\lim\limits_{x\to 1} g(x))$.

(5) [4pt] (4.2.15) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) = \sqrt{f(x)}$. If $\lim\limits_{x\to c} f$ exists, prove that $\lim\limits_{x\to c} \sqrt{f} = \sqrt{\lim\limits_{x\to c} f}$. (*Hint:* $a^2 - b^2 = (a - b)(a + b)$. Another hint: you will probably have to consider cases $\lim\limits_{x\to c} f = 0$ and $\lim\limits_{x\to c} f \neq 0$ separately.)

(6) [2pt] (5.1.7+) (Local separation from zero) Let $A \subseteq \mathbb{R}$, $c \in A$, $f : A \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that for any $\alpha \in \mathbb{R}$ such that $0 < \alpha < f(c)$, there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c) \cap A$, then $f(x) > \alpha$.

(7) [2pt] (5.1.8) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} \mid f(x) = 0\}$$

be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.

(8) [3pt] (5.1.13) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 2x$ for $x \in \mathbb{Q}$ and $g(x) = x + 3$ for $x \notin \mathbb{Q}$. Find all points at which g is continuous.